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We formulate a Dirac Hamiltonian for an electron in the field of a charged magnetic dipole, which can be solved exactly in a central field approximation. While we find no scattering resonances, we do find three categories of bound-state solutions for this Hamiltonian which may represent new electron/positron states or possible new electromagnetic composites. One category of solution is a state with very strong binding energy (greater than the electron rest mass) but with only a small amount of kinetic energy. This state corresponds to a classical orbit picture in which the electric and magnetic forces are opposed and are almost balanced. No bound states are found if the central magnetic dipole is uncharged. © *2000 American Institute of Physics.* [S0022-2488(00)03207-2]

I. INTRODUCTION

Although magnetic interactions between particles in atomic and molecular physics are generally small, these forces can be comparable to the electrostatic forces at distances of the order of the Compton wavelength and much larger at distances of a few Fermis. Furthermore, to our knowledge, there are no analytic (and nonperturbative) solutions available for magnetic interactions in a Dirac formulation that specifically address this interesting scale size. This leads us to the question, so far not satisfactorily answered: Can there be bound states or scattering resonances resulting from the magnetic forces? Specifically, we are interested in the electron–positron, electron–electron, and the electron–proton systems.

A number of authors have been concerned with the problem of magnetic interactions between fermions. The person who has perhaps written the most about it is Barut.[1] He has studied it within the framework of semiclassical theory, the Schrödinger method, Dirac theory, and coupled Maxwell–Dirac theory. Unfortunately, there is no practical way of rigorously studying two Fermi–Dirac particles interacting via the electromagnetic field in a nonperturbative way, so that one is left with either studying simplified models analytically or trying to solve a two-body equation, such as the Bethe–Salpeter equation with classical or expectation-value potentials, by a perturbation method.

The latter approach has been used by Wong and Becker,[2] Geiger *et al.*,[3] Spence and Vary,[4] and McNeil,[5] as well as by Barut.[1] A somewhat different approach, using other relativistic wave functions derived by means of QED, has been carried out numerically by Spence and Vary[4] and McNeil.[5]

Is there experimental evidence to support the idea of a bound or scattering state arising from magnetic interactions between particles? The most suggestive evidence appeared to come from the anomalous electron–positron spectral lines observed in superheavy-ion collision experiments carried out at GSI in Darmstadt.[6] The origin of these lines has not been explained satisfactorily, but is perhaps consistent with the production of a complex neutral particle (or resonance) which decays into e^+e^-. The abovementioned authors have all attempted to explain the experimental observations in terms of resonances between an electron and positron, and furthermore have claimed a certain degree of success. Yet, recent experimental work at Argonne National Laboratory[7] failed to confirm the GSI data and it is now generally believed that the earlier GSI

results are incorrect. Thus, the existence of some type of e^+e^- resonance does not seem to be supported experimentally.

II. SEMICLASSICAL CALCULATION OF AN ELECTRON IN THE FIELD OF A CHARGED MAGNETIC DIPOLE

Before we proceed to the Dirac equation, it is instructive to look at the problem semiclassically. The classical Hamiltonian for the particle (an electron) in the central field of a charged magnetic dipole is (we assume an attractive Coulomb force)

$$H=\sqrt{c^2(\mathbf{p}-e\mathbf{A}/c)^2+m^2c^4}-e^2/r. \tag{1}$$

We use spherical coordinates and shall assume $p_\theta=0$; in fact, we shall shortly restrict the discussion to orbits in the equatorial plane. If there were no magnetic field and the situation were nonrelativistic, p_ϕ (the angular momentum around the the axis of the central dipole) would be $mr^2\sin^2\theta\dot{\phi}$. For our case p_ϕ is given by

$$p_\phi=\gamma mr^2\sin^2\theta\,\dot{\phi}+r\sin\theta(e/c)\,A_\phi. \tag{2}$$

The Hamiltonian (1) then becomes

$$H=\sqrt{c^2p_r^2+\frac{c^2}{r^2\sin^2\theta}(p_\phi-r\sin\theta\,(e/c)\,A_\phi)^2+m^2c^4}-\frac{e^2}{r} \tag{3}$$

with

$$A_\phi=\mu_0\sin\theta/r^2. \tag{4}$$

Consistent with our approximation that p_θ is zero, we take $\sin\theta=1$.

Since $\partial H/\partial\phi=0$, $\dot{p}_\phi=0$, and $p_\phi=$constant. We take

$$p_\phi=\kappa\hbar, \tag{5}$$

where κ is the angular momentum quantum number. Solving (2) for $\beta_\phi\equiv r\sin\theta\dot{\phi}/c$, converting to r in units of the Compton wavelength, and defining

$$\mu=\mu_0/2\,\mu_B, \tag{6}$$

where μ_B is the Bohr magneton, we get

$$\gamma\beta_\phi=\kappa/r-\alpha\mu/r^2. \tag{7}$$

Here α is the fine structure constant. We can write the energy $\varepsilon=H/mc^2$ in the same dimensionless units,

$$\varepsilon=\sqrt{\gamma^2\beta_r^2+\gamma^2\beta_\phi^2+1}-\alpha/r, \tag{8}$$

where $\beta_r=\dot{r}/c$.

$\beta_r=0$ for circular orbits and at the r-limits of ellipical orbits, so we set this to zero here and look for bound states for the problem defined by Eqs. (7) and (8). The most strongly bound orbits are the circular orbits with a very small β_ϕ. From the centripetal force equation

$$\gamma\beta_\phi^2=\alpha/r-\alpha\beta_\phi\mu/r^2 \tag{9}$$

and Eq. (7), we find that $\beta_\phi \approx \alpha/\kappa$. Then solving Eq. (7) for r, we obtain two solutions; for the more tightly-bound solution, approximately,

$$r = \alpha\mu/\kappa \quad \text{and} \quad \varepsilon = 1 - \kappa/\mu. \tag{10}$$

The second solution has $r \approx \kappa^2/\alpha$ and has energy in the hydrogen atom range.

The interesting new solution here is the tighly-bound solution, Eq. (10). It has the following properties:

(1) The angular momentum of the bound state is supplied almost entirely by the magnetic angular momentum, allowing for very little angular velocity and thus very little kinetic energy. The binding energy is thus almost all electrostatic. Note that the electron is held in its orbit by opposed electric and magnetic forces which almost balance.

(2) It is easy to demonstrate that the circular orbit solution is stable. With an attractive coulomb force and a repulsive magnetic force, the orbit is clearly stable in the r coordinate. The only place where there might be a problem is with respect to the coordinate θ. We start with the Hamiltonian, Eq. (3), add a term in p_θ^2 and keep all the the θ dependence, but because the velocities are very small we can use a nonrelativistic expansion for H. Now

$$\partial H/\partial\theta = -\dot{p}_\theta \tag{11}$$

and noting that p_ϕ is a constant, we obtain, in the same dimensionless units we used before

$$\frac{\dot{p}_\theta}{\cos\theta} = \frac{\alpha}{r^2 \sin\theta}\left[\frac{2\alpha\mu}{r}\left(\kappa - \frac{\alpha\mu\sin^2\theta}{r}\right) + \frac{(\kappa - \alpha\mu\sin^2\theta/r)^2}{\sin^2\theta}\right]. \tag{12}$$

But $r \approx \alpha\mu/\kappa$. Therefore,

$$\frac{\dot{p}_\theta}{\cos\theta} = \frac{\alpha}{r^2 \sin\theta}\left[2\kappa^2(1-\sin^2\theta) + \frac{\kappa^2(1-\sin^2\theta)^2}{\sin^2\theta}\right] \tag{13}$$

is positive, and the circular orbit at $\theta = \pi/2$ is stable.

(3) If we change the sign of the Coulomb term to produce a repulsive interaction, we do not get any bound states classically. We do however, get a positive-energy state (resonance) with a negative binding of one or more mc^2 where the electric and magnetic forces again almost balance and the kinetic energy is very small. This state is separated from the asymtotic region by a large ''potential barrier.''

III. DIRAC ELECTRON IN A CHARGED-DIPOLE MAGNETIC FIELD

In this section we solve the Dirac equation for an electron moving in the field of a stationary, charged magnetic dipole in a similar approximation to that which we used in the semiclassical problem. The Dirac equation is,

$$[E - e\varphi + \boldsymbol{\alpha}\cdot(c\mathbf{p} - e\mathbf{A}) + \beta mc^2]\Psi = 0, \tag{14}$$

where E is the energy. We assume that the magnetic potential is produced by a point dipole μ_0 located at the origin so the $\mathbf{A}$ (only A_ϕ in spherical coordinates) is given by Eq. (4). φ is the scalar potential due to the electric charge, $e\varphi = \pm e^2/r$.

The full solution to the Dirac equation with this vector potential [Eq. (4)] is a more complicated problem than the corresponding hydrogen atom problem. We propose, however, to solve it in a ''central field approximation.'' That is, we assume that A_ϕ is a function of r only, with the angular dependence replaced by a fixed value, namely, $\langle \sin\theta \rangle$ which is of order 1. In other words,

we are restricting our analysis to solutions which tend to favor large θ-values, i.e., those with orbital angular momentum. This approximation is consistent with our semiclassical model, which involves circular orbits.

Paralleling Schiff's[8] treatment of the Dirac equation in a central field, we write

$$\boldsymbol{\alpha}\cdot\mathbf{p}-(e/c)\boldsymbol{\alpha}\cdot\mathbf{A}=\alpha_r p_r+i\hbar\alpha_r\beta\kappa/r-i(e/c)\alpha_r\beta\mu_0\langle\sin\theta\rangle/r^2, \tag{15}$$

where $\kappa=j+\frac{1}{2}$ is the angular momentum quantum number. We choose a representation in which both the Hamiltonian and κ are diagonal,

$$\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \text{ and } \alpha_r=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}. \tag{16}$$

Then spinor Ψ has two components which may be written

$$\Psi=\begin{pmatrix}F(r)/r\\ G(r)/r\end{pmatrix} \tag{17}$$

and since

$$p_r=-i\hbar\,(\partial/\partial r+1/r), \tag{18}$$

we obtain the equations

$$\begin{aligned}(\epsilon+1-\eta/r)\,F(r)-G'(r)-(\kappa/r-\alpha\,\mu/r^2)\,G(r)=0,\\ (\epsilon-1-\eta/r)\,G(r)+F'(r)-(\kappa/r-\alpha\,\mu/r^2)\,F(r)=0,\end{aligned} \tag{19}$$

where now r is in units of the Compton wavelength ($\lambda_c=\hbar/mc$), ϵ is in units of mc^2, α is the fine structure constant, $\eta=\pm\,\alpha$ ($+$ for repulsive potential, $-$ for attractive), and μ is a dimensionless magnetic moment (in units of the Bohr magneton, μ_B). Specifically,

$$\mu=\mu_0\langle\sin\theta\rangle/2\mu_B\,. \tag{20}$$

Barut[9] had obtained one solution to Eqs. (19) for the case where $\eta=0$ (i.e., no Coulomb interaction). His solution,

$$F(r)=0,\;\; G(r)=(1/r^{\kappa})\exp[-\alpha\mu/r],\;\; \epsilon=1. \tag{21}$$

This result suggested that there might be other solutions in the vicinity of $\epsilon=1$ (i.e., a small amount of binding, or a positive energy resonance) when the Coulomb interaction is included. This turns out not to be the case, and it appears that Barut's solution (with only one non-zero component of the spinor Ψ) is an artifact.

We now proceed to solutions of Eqs. (19). We assume the spinor components can be represented in the form

$$\begin{aligned}F(r)=\exp[-(\alpha\,\mu+\omega\,r^2)/r]\,f(r)\,r^{-\kappa+1},\\ G(r)=\exp[-(\alpha\,\mu+\omega\,r^2)/r]\,g(r)r^{-\kappa},\end{aligned} \tag{22}$$

where $f(r)$ and $g(r)$ are found to be truncated power series in r. For each value of κ, we have found several solutions corresponding to differing numbers of terms in the series. In all cases, we find that

$$\omega=\sqrt{1-\epsilon^2}. \tag{23}$$

TABLE I. Coefficients for $f(\rho)$ and $g(\rho)$ in Eq. (26) for the two lowest order solutions.

f_0	f_1	f_2	g_0	g_1	g_2
1	0	0	$-2b/\eta$	$-2b/\eta+\eta$	0
1	$\frac{-(-2b+\eta^2-\eta x_1)}{\eta(\eta-2x_1)}$	0	$-2b/\eta$	$-2b/\eta+\eta$	$\frac{-x_1(-2b+\eta^2-\eta x_1)}{\eta(\eta-2x_1)}$

These are bound-state solutions with energies between $\pm mc^2$. Because Eq. (23) can be written as $\omega=i\sqrt{\epsilon^2-1}$ there could possibly be oscillatory solutions corresponding to energies greater than mc^2, but we find no such solutions for real μ. Furthermore, there are not solutions for $\omega=0$ except for the one mentioned earlier [Eq. (21)].

Since all of our physically meaningful solutions correspond to energies between $\pm mc^2$ it is convenient to make a change of variable in Eq. (19). We transform to $\rho=(r/\lambda_c)\sqrt{1-\epsilon^2}$ and $b=\alpha\mu\sqrt{1-\epsilon^2}$. This produces a substantial simplification in the subsequent algebra. We obtain for Eq. (19),

$$\begin{aligned} F'(\rho)+(-\kappa/\rho+b/\rho^2)\,F(\rho)-(1/x_1+\eta/\rho)\,G(\rho)=0,\\ G'(\rho)+(\kappa/\rho-b/\rho^2)\,G(\rho)-(x_1-\eta/\rho)\,F(\rho)=0, \end{aligned} \tag{24}$$

where $x_1=(1+\epsilon)/\sqrt{1-\epsilon^2}$. Clearly,

$$\epsilon=(x_1^2-1)/(1+x_1^2). \tag{25}$$

We write the solution in terms of ρ as

$$\begin{aligned} F(\rho)=-\rho^{-\kappa+1}\,f(\rho)\exp[-(b+\rho^2)/\rho],\\ G(\rho)=\rho^{-\kappa}\,g(\rho)\exp[-(b+\rho^2)/\rho], \end{aligned} \tag{26}$$

where $f(\rho)=1+f_1\rho+f_2\rho^2+\cdots$ and $g(\rho)=g_0+g_1\rho+g_2\rho^2+\cdots$ are truncated power series and $g(\rho)$ has one more term in its series than does $f(\rho)$.

We start with the simplest case, $f(\rho)=1, g(\rho)=g_0+g_1\rho$, where the solution can be written in simple terms,

$$\eta x_1=\kappa-1\pm\sqrt{(\kappa-1)^2+\eta^2},\quad 2b=\eta^2-\eta x_1,\quad g_0=-2\,b/\eta,\quad \text{and}\quad g_1=-2\,b/\eta+\eta.$$

Note that there are two values of b for each x_1, one for $\eta=+\alpha$, and one for $\eta=-\alpha$. As we add terms to the series, we generate new solutions. It can be shown that if the $f(\rho)$ series has n terms then

$$\eta x_1=(\kappa-n)\pm\sqrt{(\kappa-n)^2+\eta^2}. \tag{27}$$

However, the corresponding expressions for b become successively more complicated beyond $n=1,2$. Table I displays the coefficients in the first two truncated power series and Table II, the values of b and ηx_1 for the two series. Notice that there are five distinct roots for the second series, the first two of which are identical to those of the first series.

The above solutions are not the only analytic solutions to Eq. (24). The central magnetic dipole can be oriented in the opposite direction and this changes the sign of b in Eq. (24). This procedure generates another family of solutions with

$$\eta x_1=-(\kappa+n)\pm\sqrt{(\kappa+n)^2+\eta^2}. \tag{28}$$

TABLE II. Expressions for b and ηx_1 for the two lowest order solutions to Eqs. (24), $rt_1=\sqrt{\eta^2+(\kappa-1)^2}$ and $rt_2=\sqrt{\eta^2+(\kappa-2)^2}$.

b	ηx_1
$\frac{1}{2}(1+\eta^2-\kappa-rt_1)$	$-1+\kappa+rt_1$
$\frac{1}{2}(1+\eta^2-\kappa+rt_1)$	$-1+\kappa-rt_1$
$\frac{1}{2}(3+\eta^2-2\kappa-rt_2+\sqrt{(\kappa-3)^2+2\eta^2-4rt_2})$	$-2+\kappa+rt_2$
$\frac{1}{2}(3+\eta^2-2\kappa+rt_2-\sqrt{(\kappa-3)^2+2\eta^2-4rt_2})$	$-2+\kappa-rt_2$
$\frac{1}{2}(3+\eta^2-2\kappa+rt_2+\sqrt{(\kappa-3)^2+2\eta^2-4rt_2})$	$-2+\kappa-rt_2$

There is a relationship between these solutions and the previous family of solutions; each new solution is obtained from a previous one by simultaneously changing $F\rightarrow G$, $G\rightarrow F$, $\kappa\rightarrow-\kappa$, $x_1\rightarrow 1/x_1$ and attractive Coulomb potential $\leftrightarrow$ repulsive Coulomb potential.

From the numerical evaluations, we find that many of the ''solutions'' must be discarded. Some correspond to complex or negative b values. Others correspond to unphysically large values of μ. The physically acceptable solutions fall into three categories; a group at $\epsilon=0$ with b (or μ values) of the order of one Bohr magneton, and a group (mostly with attractive Coulomb interaction) clustering near $\epsilon=1$ (i.e., mc^2) with magnetic moments in the fractional Bohr magneton range, and a group (mostly with repulsive Coulomb interaction) near $\epsilon=-1$, again in the fractional Bohr magneton range.

A few representative evaluations for some physically acceptable solutions are presented in Table III for κ values of 2, 1, -1, -2. These were generated from Eqs. (27) and (28) with $n=1$ and 2. Some additional values for energies between 0 and -1 have been obtained by numerical integration of the differential equations. We note that the range of μ values for which a solution of the equations exists is quite limited for each value of κ.

IV. PROPERTIES OF THE WAVE FUNCTIONS

A few remarks about the wave function are in order. The $\exp(-b/\rho)$ factor only affects the wave function at very small ρ; the distant asymptotic region is governed primarily by the $\exp(-\rho)$ factor. Furthermore, in the asymptotic region $G\approx -x_1F$.

TABLE III. Solution parameters of the Dirac Hamiltonian for an electron in the field of a magnetic dipole μ with charge $\pm e$.

Attractive Coulomb Potential				
κ	x_1	b	ϵ	μ
1	1	0.003676	0	0.502
1[a]	0.5407	0.002	−0.5473	0.327
1[a]	0.2667	0.001	−0.8673	0.275
2	1	0.0109	0	1.5
2[a]	0.5	0.0054	−0.6	0.925
2	0.003649	0.0000399	−0.99997	0.75
−1	274	0.00003215	0.99997	0.604
1	548	0.00001997	0.999993	0.75
1	822	0.0000133	0.999997	0.75
2	822	0.0000222	0.999997	1.25

Repulsive Coulomb Potential

There are an equal number of repulsive Coulomb solutions with a one-to-one correspondence to the analytic solutions found above. The relationship between solutions is $F\rightarrow G$, $G\rightarrow F$, $\kappa\rightarrow-\kappa$, $x_1\rightarrow 1/x_1$, and attractive Coulomb $\rightarrow$ repulsive.

[a]Indicates solutions obtained by numerical integration.

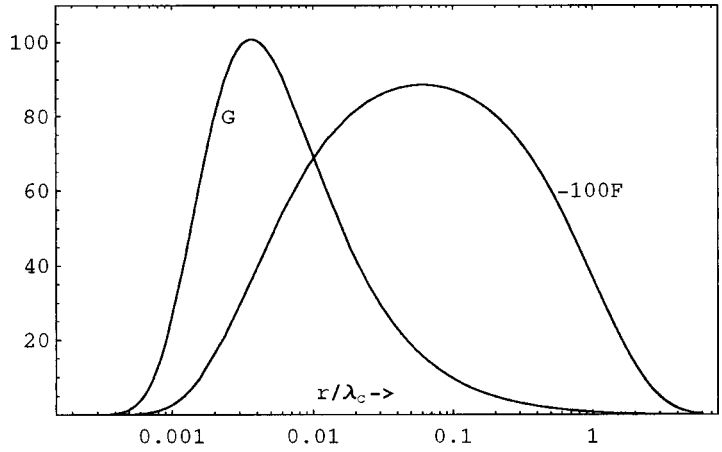

FIG. 1. The spinor components, F and G, for the Dirac Hamiltonian solution for the case with $\epsilon=0$, $\kappa=1$ (Table III, first entry). Note that the peak in G occurs at the position of the circular orbit of semiclassical theory, namely, $r/\lambda_c = \alpha\,\mu/\kappa$.

Consider the cases from Table III with $\epsilon=0$; because of the $\exp(-\rho)$ factor these two solutions extend out to the order of the Compton wavelength. A plot of the two spinor components, F and G for the $\epsilon=0,\kappa=1$, (attractive) solution are shown in Fig. 1. Notice that the wave function is strongly confined to a distance of less than a Compton wavelength, and that the peak (in G, for example) occurs even further in, at $\rho=b$.

The solutions for $\kappa=2$ with $\epsilon=0$ and -0.99997 show a similar behavior, both having peaks in G at $\rho=b/2$. In fact, other solutions corresponding to the *attractive* Coulomb potential show similar behavior, with a peak in G (or a principal peak in G) at $\rho=b/\kappa$. Because both ρ and b contain the factor $\sqrt{1-\epsilon^2}$ in their definitions, the peak in G occurs at $r/\lambda_c=\alpha\mu/\kappa$, i.e., at a few Fermis. Note that these are the r-values of the classical orbits found in Sec. I. All of these solutions correspond to cases where $l\neq 0$, and thus correspond to solutions where the wave function is concentrated at large θ.

Equations (24) were integrated numerically for a number of ϵ values between 0 and -1, and the results generated solutions for other values of μ (see Table III). These solutions were similar in form to the analytic solutions discussed above and provided a range of μ values of about a factor of 2 for each value of κ.

The second group of eigenstates, which are clustered near $\epsilon=1$ and correspond in most cases to an attractive Coulomb potential is entirely different from the above. Because $\rho = (r/\lambda_c)\sqrt{1-\epsilon^2}$ these states extend out to or beyond the Bohr radius. Let us look now at the spinor components for the case $\kappa=-1$, $\epsilon=+0.99997$, which we display in Fig. 2. The peaks in F and G occur near $\rho=1$ which in this case occurs at $r=137\lambda_c$, i.e., at the Bohr radius. There is evidence of a peak associated with b, but it is very small compared to the main peak at $\rho=1$. And except for the $\exp(-b/\rho)$ factor (which does not play an important role) the wave function looks like a hydrogen atom solution; in fact, the binding energy of the state is $-(\alpha^2/2)mc^2$. It would appear that in this group of states the magnetic interaction does not play a major role in binding, but instead acts as a perturbation on the Coulomb energy solutions. This category of solutions can also be extended by numerical integration to provide solutions for other values of μ; however, the range of allowed energies is very small, the x_1 values varying by less than 0.1% for μ values less than 10. These solutions cannot be extended to join the category 1 solutions; note that they satisfy a different differential equation (namely, one with b replaced by $-b$).

What about the solutions corresponding to a *repulsive* Coulomb interaction? For these solutions the F-function dominates (i.e., is much larger than G,) and most cases correspond to negative κ and cluster in energy near $\epsilon=-1$. Consider the solution for $\kappa=1$, $\epsilon=-0.99997$, the *repulsive* analog of the solution shown in Fig. 2. How is it that an eigenstate that is concentrated in the Bohr-radius region with a modest magnetic dipole interaction can have such a strong

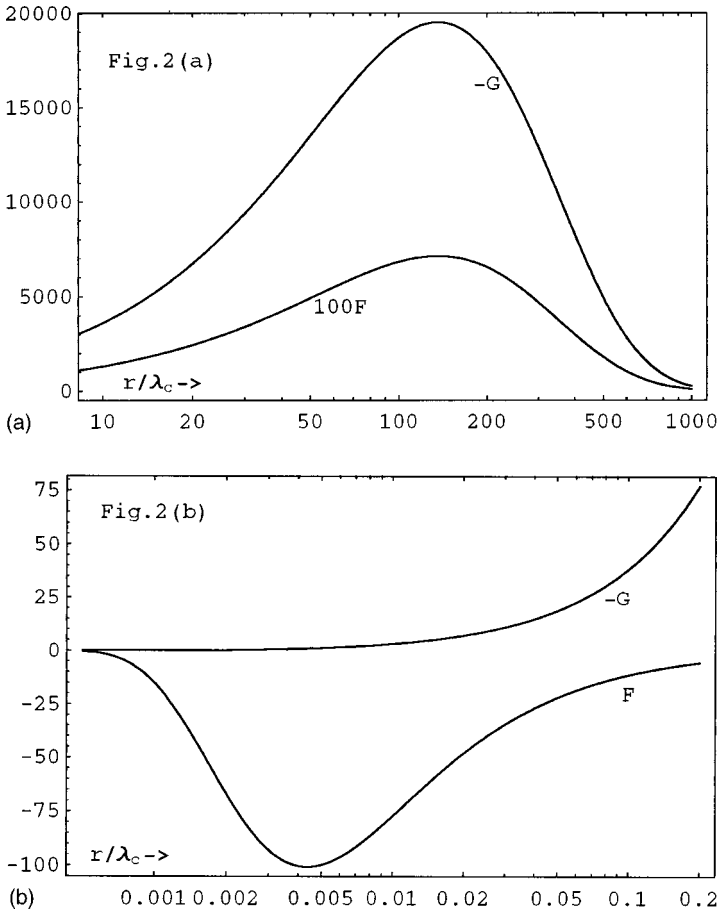

FIG. 2. The spinor components for the Dirac Hamiltonian solution for the case with $\epsilon=0.99997$, $\kappa=-1$ at the Bohr scale (a) and the Fermi scale (b). Note that the wave function is peaked out at the Bohr radius. This wave function is the attractive analog of a solution using $\kappa=1$ and line 3 of Table II.

binding, $E\approx -mc^2$? The answer, we believe, is that these repulsive Coulomb states correspond to something else—they are associated with the negative energy states of the Dirac equation and represent attractive Coulomb interaction for these states, with energies measured with respect to $-mc^2$. To help understand this we combine the two equations (19) into one equation for G and get a Schrodinger-type equation; we find the dominant contributions to the $E-V_{\text{eff}}(r)$ term are

$$\epsilon^2-1-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\,\alpha\,\mu\,(\kappa+1)}{r^3}-\frac{\alpha^2\,\mu^2}{r^4}\pm\frac{2\,\alpha\,\epsilon}{r}. \tag{29}$$

We note several things: (1) κ and μ must change sign together in order to maintain the sign of the magnetic term $(1/r^3)$, (2) if we replace ϵ by $-\epsilon$ we get the same solution provided we change the sign of the Coulomb term.

Finally, it is instructive to calculate the average velocity for a typical solution to our Dirac Hamiltonian. The average velocity in the ϕ direction is given by $\langle v_\phi\rangle=c\langle\alpha_\phi\rangle$ and, $\alpha_\phi=i\,\alpha_r\beta$. The calculated velocities are all $0.05c$ or less, and interestingly, not even close to relativistic.

V. SUMMARY OF RESULTS

Our solutions only partially support the results and speculations of Barut,[1] who conjectured that magnetic interactions should play a dominant role in the binding of Fermi–Dirac particles (electrons, protons, etc.) at small distances. In most case Barut worked with the second-order

differential equation and an effective potential; here the ''magnetic potential'' $(1/r^3)$ term is quite large and negative at short range and the kinetic energy is also large. But Barut does not carry his results through numerically for a specific particle system. Now, we do find bound states of an electron in the field of a charged magnetic dipole, but the range of parameters is much more limited than one would expect from Barut's publications. For example, we do not find solutions of our Dirac Hamiltonian for the case where the central dipole magnetic moment is very small (e.g., a few nuclear magnetons). Therefore, these results do not appear to apply to the electron–proton system. We have not found any solutions either with the magnetic dipole having zero charge (i.e., no electric field). In addition we do not find any scattering resonance states within the solution set for this Hamiltonian (although, classically, we expect a scattering resonance when the Coulomb term is repulsive).

We do, however, find physically acceptable solutions when the central dipole magnetic moment is of the order of one Bohr magneton, so that these results could represent the $e^+ e^-$ system. But there are some difficulties with this interpretation; the electron and positron are not treated on an equal footing, and there is no orbit-orbit interaction in our model. Interestingly, although the binding energy of these states is quite large, the kinetic energy is not large at all, i.e., not relativistic. Therefore, the expected retardation effects in a more exact theory may not play such an important role in these states.

The problem we have solved is that of an electron in the field of a heavy charged particle with a magnetic moment; however, the magnetic moment of the heavy particle is large—of the order of that of the electron. Furthermore, there is a simple classical picture for this Dirac problem—the electron is held in its orbit by electric and magnetic forces, and in the strongly–bound cases the forces are opposed and almost balanced. This solution cannot be simply extended to the equal mass, relativistic electron–positron system, but since the kinetic energy of this solution is very small the results are suggestive that there might be a tightly-bound state for the electron and positron where part of the forces are magnetic in origin.

In this paper, we have exhibited a number of analytic solutions to our postulated Dirac–Hamiltonian. In addition, there are solutions corresponding to other values of μ for the energy range 0 to $-mc^2$ which are obtained by numerical integration of the differential equations so that we are not limited to the precise μ values of the analytic solutions. The most interesting solutions that we find are these strongly-bound states, but the second group of eigenstates clustering around $\epsilon=1$ deserve to be studied more fully. Do they admit solutions (by numerical integration) for much smaller magnetic moments, and do they go over to hydrogen-atom type solutions? We should also mention the analytic solutions we found for very large magnetic moments, and which we rejected as unphysical. It is possible that these solutions could be extended by numerical integration of the differential equations to cases with smaller (but still large) magnetic moments. They might correspond to particles yet to be discovered.

More investigations are required, not only to determine if there are other solutions to this Dirac Hamiltonian, but also to establish the meaning of the solutions which we have found. Do they represent bound states of the electron and positron or could they perhaps represent a ''composite'' model of some ''fundamental'' particle? With appropriate change of scale, the results obtained can be applied to other Fermi–Dirac particles. The solutions can also be extended to the case where the central dipole has a charge greater than the electron charge; but if its charge is increased, then the magnetic moment must increase also.

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APPENDIX

Our Dirac equation, Eq. (19) can be obtained from another situation, namely, from the interaction of the anomalous magnetic moment of the electron with the Coulomb field of a central particle.[10] The central particle does not have to have a magnetic moment.

We add the term $i\mu\mu_B\gamma^\mu\gamma^\nu F_{\mu\nu}$ to the covariant form of Eq. (14), where μ is now the dimensionless magnetic moment correction. Keeping only those $F_{\mu\nu}$ components which relate to the electric field, we find that Eq. (14) reduces to Eq. (19).

[1] See, e.g., A. O. Barut,''Lectures on magnetic interactions of stable particles and magnetic resonances'' in *Group Theory and Its Applications in Physics*, edited by T. H. Seligman (AIP, New York, 1980), pp. 73–108; A. O. Barut, Z. Phys. A **336**, 317 (1990).
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